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New Hermite–Hadamard-type inequalities for convex functions (I)

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ABSTRACT

In this work we establish some new Hermite–Hadamard-type inequalities for convex functions and give several applications for special means.

1. Introduction

Throughout this work, let $f : [a, b] \to \mathbb{R}$ be convex, $a \le x < y \le y' < x' \le b$, $x + x' = y + y'$ and $\Omega = [x, y] \cup [y', x']$. We define the following functions on $[0, 1]$ that are associated with the well known Hermite–Hadamard inequality [1]

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(s)\,ds \le \frac{f(a)+f(b)}{2}, \tag{1.1}$$

namely

$$H(t) = \frac{1}{b-a}\int_a^b f\left(ts + (1-t)\frac{a+b}{2}\right) ds;$$

$$H_1(t) = \frac{1}{2(y-x)}\int_x^y \left[f(ts + (1-t)y) + f\left(t\left(y + y' - s\right) + (1-t)y'\right)\right] ds;$$

$$H_2(t) = \frac{1}{2(y-x)}\int_x^y \left[f\left(ts + (1-t)y'\right) + f\left(t\left(y + y' - s\right) + (1-t)y\right)\right] ds;$$

$$F(t) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f(ts + (1-t)u)\,dsdu;$$

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$$F_1(t) = \frac{1}{4(y-x)^2}\int_{\Omega}\int_{\Omega} f(ts + (1-t)u)\,dsdu;$$

$$P(t) = \frac{1}{2(b-a)}\int_a^b \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)s\right) + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)s\right)\right] ds$$

and

$$P_1(t) = \frac{1}{2(y-x)}\int_x^y \left[f(tx + (1-t)s) + f\left(tx' + (1-t)\left(x + x' - s\right)\right)\right] ds.$$

Remark 1. We note that $\Omega = [a, b]$ and $H(t) = H_1(t) = H_2(t)$, $F(t) = F_1(t)$, $P(t) = P_1(t)$ on $[0, 1]$ as $x = a$, $y = y' = \frac{a+b}{2}$ and $x' = b$.

For some results which generalize, improve, and extend this famous integral inequality (1.1) see [2–6].

Dragomir [2] established the following Hermite–Hadamard-type inequalities related to the functions H, F which refine the first inequality of (1.1).

Theorem A. *Let f, H be defined as above. Then H is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have*

$$f\left(\frac{a+b}{2}\right) = H(0) \le H(t) \le H(1) = \frac{1}{b-a}\int_a^b f(s)ds. \tag{1.2}$$

Theorem B. *Let f, F be defined as above. Then:*

(1) *F is convex on $[0, 1]$, symmetric about $\frac{1}{2}$, decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$, and we have*

$$\sup_{t\in[0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a}\int_a^b f(s)ds$$

and

$$\inf_{t\in[0,1]} F(t) = F\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{s+u}{2}\right) dsdu.$$

(2) *We have*

$$f\left(\frac{a+b}{2}\right) \le F\left(\frac{1}{2}\right); \qquad H(t) \le F(t), \quad t \in [0, 1]. \tag{1.3}$$

Yang and Hong [3] established the following Hermite–Hadamard-type inequality related to the function P and which refines the second inequality of (1.1).

Theorem C. *Let f, P be defined as above. Then P is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have*

$$\frac{1}{b-a}\int_a^b f(s)ds = P(0) \le P(t) \le P(1) = \frac{f(a) + f(b)}{2}. \tag{1.4}$$

In this work, we shall establish some new Hermite–Hadamard-type inequalities which generalize Theorems A–C and give several applications.

2. The main results

In order to prove our main results, we need the following lemmas:

Lemma 1 (*See [4]*)**.** *Let f be defined as above and let $a \le A \le C \le D \le B \le b$ with $A + B = C + D$. Then*

$$f(C) + f(D) \le f(A) + f(B).$$

The assumptions in Lemma 1 can be weakened as in the following lemma:

Lemma 2. *Let f be defined as above and let $a \le A \le C \le B \le b$ and $a \le A \le D \le B \le b$ with $A + B = C + D$. Then*

$$f(C) + f(D) \le f(A) + f(B).$$

Now, we are ready to state and prove our new results.

Theorem 1. *Let $x, y, y', x', \Omega, f, H_1$ be defined as above. Then:*

(1) *H_1 is convex on $[0, 1]$.*
(2) *H_1 is increasing on $[0, 1]$ and the following inequalities hold for all $t \in [0, 1]$:*

$$\frac{f(y)+f\left(y'\right)}{2} = H_1(0) \le H_1(t) \le H_1(1) = \frac{1}{2\,(y-x)} \int_{\Omega} f(s)ds. \tag{2.1}$$

$$\begin{aligned} H_1(t) &\le t \cdot \frac{1}{2\,(y-x)} \int_{\Omega} f(s)ds + (1-t) \cdot \frac{f(y)+f\left(y'\right)}{2} \\ &\le \frac{1}{2\,(y-x)} \int_{\Omega} f(s)ds \le \frac{f(x)+f\left(x'\right)}{2}. \end{aligned} \tag{2.2}$$

Proof. (1) It is easily observed from the convexity of f that H_1 is convex on $[0, 1]$.

(2) Let $t_1 < t_2$ in $[0, 1]$. By Lemma 2, the inequality

$$f\,(t_1 s + (1-t_1)\,y) + f\left(t_1\left(y+y'-s\right) + (1-t_1)\,y'\right) \le f\,(t_2 s + (1-t_2)\,y) + f\left(t_2\left(y+y'-s\right) + (1-t_2)\,y'\right)$$

holds for all $s \in [x, y]$. Integrating the above inequality over s on $[x, y]$, and dividing both sides by $2\,(y-x)$, we have $H_1\,(t_1) \le H_1\,(t_2)$. Thus, H_1 is increasing on $[0, 1]$ and the inequality (2.1) holds. Using the convexity of f, the inequality (2.1) and the substitution rule for integration, we obtain the first and second inequalities of (2.2). Using simple techniques of integration, we have the following identity:

$$\frac{1}{2\,(y-x)} \int_{\Omega} f(s)ds = \frac{1}{2\,(y-x)} \int_{x}^{y} \left[f(s) + f\left(y+y'-s\right)\right] ds.$$

By Lemma 2, the inequality

$$f(s) + f\left(y+y'-s\right) \le f(x) + f\left(x'\right)$$

holds for all $s \in [x, y]$. Integrating the above inequality over s on $[x, y]$, dividing both sides by $2\,(y-x)$ and using the above identities, we derive the last inequality of (2.2). This completes the proof. □

Theorem 2. *Let $x, y, y', x', \Omega, f, H_2$ be defined as above. Then:*

(1) *H_2 is convex on $[0, 1]$.*
(2) *The following inequalities hold for all $t \in [0, 1]$:*

$$\begin{aligned} f\left(\frac{y+y'}{2}\right) &\le H_2(t) \\ &\le t \cdot \frac{1}{2\,(y-x)} \int_{\Omega} f(s)ds + (1-t) \cdot \frac{f(y)+f\left(y'\right)}{2} \\ &\le \frac{1}{2\,(y-x)} \int_{\Omega} f(s)ds. \end{aligned} \tag{2.3}$$

$$H_2(t) \le H_1(t). \tag{2.4}$$

Proof. (1) It is easily observed from the convexity of f that H_2 is convex on $[0, 1]$.

(2) Using the convexity of f, the inequality (2.1) and the substitution rule for integration, we obtain the inequality (2.3). Using Lemma 2, the inequality

$$f\left(ts + (1-t)\,y'\right) + f\left(t\left(y+y'-s\right) + (1-t)\,y\right) \le f\,(ts + (1-t)\,y) + f\left(t\left(y+y'-s\right) + (1-t)\,y'\right)$$

holds for all $t \in [0, 1]$ and $s \in [x, y]$. Integrating the above inequality over s on $[x, y]$, dividing both sides by $2\,(y-x)$ and using the definitions of H_1 and H_2, we derive the inequality (2.4). This completes the proof. □

Theorem 3. *Let $x, y, y', x', \Omega, f, H_1, P_1$ be defined as above. Then we have the following results:*

(1) *P_1 is convex on $[0, 1]$.*
(2) *P_1 is increasing on $[0, 1]$ and the following inequalities hold for all $t \in [0, 1]$:*

$$\frac{1}{2(y-x)}\int_{\Omega} f(s)ds = P_1(0) \le P_1(t) \le P_1(1) = \frac{f(x)+f\left(x'\right)}{2}. \tag{2.5}$$

$$\begin{aligned} P_1(t) &\le (1-t)\cdot\frac{1}{2(y-x)}\int_{\Omega} f(s)ds + t\cdot\frac{f(x)+f\left(x'\right)}{2} \\ &\le \frac{f(x)+f\left(x'\right)}{2}. \end{aligned} \tag{2.6}$$

$$H_1(t) \le P_1(t) \quad (t \in [0, 1]). \tag{2.7}$$

Proof. (1) It is easily observed from the convexity of f that P_1 is convex on $[0, 1]$.

(2) Let $t_1 < t_2$ in $[0, 1]$. By Lemma 2, the inequality

$$f(t_1x + (1-t_1)s) + f\left(t_1x' + (1-t_1)\left(x+x'-s\right)\right) \le f(t_2x + (1-t_2)s) + f\left(t_2x' + (1-t_2)\left(x+x'-s\right)\right)$$

holds for all $s \in [x, y]$. Integrating the above inequality over s on $[x, y]$, and dividing both sides by $2(y-x)$, we have $P_1(t_1) \le P_1(t_2)$. Thus, P_1 is increasing on $[0, 1]$ and (2.5) holds.

Using the convexity of f, the inequality (2.5) and the substitution rule for integration, the inequality (2.6) holds. Finally, the inequality (2.7) follows from the inequalities (2.1) and (2.5). This completes the proof. □

Theorem 4. *Let $x, y, y', x', \Omega, f, H_1, H_2, F_1$ be defined as above. Then we have the following results:*

(1) *F_1 is convex on $[0, 1]$ and symmetric about $\frac{1}{2}$.*
(2) *F_1 is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$,*

$$\sup_{t\in[0,1]} F_1(t) = F_1(0) = F_1(1) = \frac{1}{2(y-x)}\int_{\Omega} f(s)ds \tag{2.8}$$

and

$$\inf_{t\in[0,1]} F_1(t) = F_1\left(\frac{1}{2}\right) = \frac{1}{4(y-x)^2}\int_{\Omega}\int_{\Omega} f\left(\frac{s+u}{2}\right) dsdu. \tag{2.9}$$

(3) *We have*

$$\frac{H_1(t)+H_2(t)}{2} \le F_1(t) \quad (t \in [0, 1]) \tag{2.10}$$

and

$$\frac{f(y) + 2f\left(\frac{y+y'}{2}\right) + f\left(y'\right)}{4} \le F_1(t) \quad (t \in [0, 1]). \tag{2.11}$$

Proof. (1) It is easily observed from the convexity of f that F_1 is convex on $[0, 1]$.

On changing variables, we have $F_1(t) = F_1(1-t) \quad (t \in [0, 1])$ from which we get that F_1 is symmetric about $\frac{1}{2}$.

(2) Let $t_1 < t_2$ in $\left[0, \frac{1}{2}\right]$. Using the symmetry of F_1 and Lemma 1, we have

$$F_1(t_2) = \frac{1}{2}\left[F_1(t_2) + F_1(1-t_2)\right] \le \frac{1}{2}\left[F_1(t_1) + F_1(1-t_1)\right] = F_1(t_1).$$

From the above result, we obtain that F_1 is decreasing on $\left[0, \frac{1}{2}\right]$. Since F_1 is symmetric about $\frac{1}{2}$ and F_1 is decreasing on $\left[0, \frac{1}{2}\right]$, we get that F_1 is increasing on $\left[\frac{1}{2}, 1\right]$. Using the symmetry and monotonicity of F_1, we derive the inequalities (2.8) and (2.9).

(3) Using the substitution rules for integration, we have the following identities:

$$\begin{aligned} F_1(t) = \frac{1}{4(y-x)^2}\Bigg\{&\int_x^y\int_x^y \left[f(ts + (1-t)u) + f\left(ts + (1-t)\left(y+y'-u\right)\right)\right] dsdu \\ &+ \int_x^y\int_x^y \left[f\left(t\left(y+y'-s\right) + (1-t)u\right) + f\left(t\left(y+y'-s\right) + (1-t)\left(y+y'-u\right)\right)\right] dsdu\Bigg\} \end{aligned}$$

and

$$\begin{aligned}\frac{H_1(t)+H_2(t)}{2} &= \frac{1}{4(y-x)^2}\left\{\int_x^y\int_x^y \left[f(ts+(1-t)y)+f\left(ts+(1-t)y'\right)\right]dsdu\right.\\ &\quad\left.+\int_x^y\int_x^y\left[f\left(t\left(y+y'-s\right)+(1-t)y\right)+f\left(t\left(y+y'-s\right)+(1-t)y'\right)\right]dsdu\right\}\end{aligned}$$

for all $t\in[0,1]$. By Lemma 2, the following inequalities hold for all $t\in[0,1]$, $s\in[x,y]$ and $u\in[x,y]$:

$$f(ts+(1-t)y)+f\left(ts+(1-t)y'\right)\le f(ts+(1-t)u)+f\left(ts+(1-t)\left(y+y'-u\right)\right).$$

$$\begin{aligned}&f\left(t\left(y+y'-s\right)+(1-t)y\right)+f\left(t\left(y+y'-s\right)+(1-t)y'\right)\\ &\quad\le f\left(t\left(y+y'-s\right)+(1-t)u\right)+f\left(t\left(y+y'-s\right)+(1-t)\left(y+y'-u\right)\right).\end{aligned}$$

Dividing the above inequalities by $4(y-x)^2$, integrating them over s on $[x,y]$ and over u on $[x,y]$, and using the above identities, we derive the inequality (2.10).

Finally, using the inequalities (2.1), (2.3) and (2.10), we derive the inequality (2.11). This completes the proof. $\square$

Remark 2. Let $x=a$, $y=y'=\frac{a+b}{2}$ and $x'=b$ in Theorems 1–4. Then Theorems 1 and 2 reduce to Theorem A, Theorem 4 reduces to Theorem B and Theorem 3 reduces to Theorem C.

The following corollaries are natural consequences of Theorems 1–4.

Corollary 1. *The following inequalities hold for all $t\in[0,1]$:*

$$\begin{aligned}\frac{f(y)+f\left(y'\right)}{2} &\le H_1(t)\le\frac{1}{2(y-x)}\int_\Omega f(s)ds\\ &\le P_1(t)\le\frac{f(x)+f\left(x'\right)}{2}.\end{aligned}\tag{2.12}$$

$$\frac{1}{2}\left[f\left(\frac{y+y'}{2}\right)+\frac{f(y)+f\left(y'\right)}{2}\right]\le\frac{H_1(t)+H_2(t)}{2}\le F_1(t).\tag{2.13}$$

$$F_1(t)\le\frac{1}{2(y-x)}\int_\Omega f(s)\,ds\le P_1(t)\le\frac{f(x)+f\left(x'\right)}{2}.\tag{2.14}$$

$$\frac{1}{2}\left[f\left(\frac{y+y'}{2}\right)+\frac{f(y)+f\left(y'\right)}{2}\right]\le\frac{H_1(1-t)+H_2(1-t)}{2}\le F_1(t).\tag{2.15}$$

Corollary 2. *Using Corollary 1 applied to the convex functions*

(1) s^p $(s\in(0,\infty), p\in(-\infty,0)\cup[1,\infty))$,
(2) $-s^p$ $(s\in(0,\infty), p\in(0,1))$,
(3) s^{-1} $(s\in(0,\infty))$,
(4) $-\ln s$ $(s\in(0,\infty))$,
(5) $s\ln s$ $(s\in(0,\infty))$,

we may obtain several inequalities for the arithmetic, harmonic, logarithmic and p-logarithmic means. The details are omitted.

Conjecture 1. *Theorems 1–4 also hold for Wright-convex functions* [7].

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